

Lecture 14: Compatible Observables, Uncertainty, and the Delta Well

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Lecture 13 obtained the complete harmonic-oscillator energy spectrum and its Gaussian ground state. In this lecture we finish the oscillator eigenfunctions, introduce degeneracy and common eigenstates, derive the general uncertainty relation, and solve the bound-state problem for an attractive delta-function well.

Questions.

- How does the raising operator generate excited oscillator states?
- What does it mean for an eigenvalue to be degenerate?
- What does a measurement determine when its eigenvalue is degenerate?
- How are common eigenstates related to commutators?
- How does the commutator lead to the general uncertainty relation?
- Why does an attractive delta-function well have exactly one bound state?

Lecture roadmap. We first apply the raising operator to the oscillator ground state. We then study degenerate eigenspaces and simultaneous eigenstates. Next we use the Cauchy–Schwarz inequality to derive the uncertainty relation. Finally, we determine the matching condition, bound-state energy, and normalized wave function of the attractive delta well.

Review from Lecture 13

For the harmonic oscillator,

$$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = +\hbar\omega\hat{a}^\dagger. \quad (1)$$

Therefore, when the result is nonzero,

$$\hat{a}\phi_E \propto \phi_{E-\hbar\omega}, \quad \hat{a}^\dagger\phi_E \propto \phi_{E+\hbar\omega}. \quad (2)$$

The bottom of the ladder satisfies

$$\hat{a}\phi_0 = 0, \quad E_0 = \frac{1}{2}\hbar\omega, \quad \phi_0(x) = Ae^{-x^2/x_0^2}. \quad (3)$$

The allowed energies are

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots \quad (4)$$

1 Completing the oscillator eigenfunctions

1.1 The ground-state equation

The ground-state condition is the first-order equation

$$\left(\frac{d}{dx} + \frac{2x}{x_0^2} \right) \phi_0(x) = 0. \quad (5)$$

Separating variables gives

$$\frac{1}{\phi_0} \frac{d\phi_0}{dx} = -\frac{2x}{x_0^2}, \quad (6)$$

and hence

$$\boxed{\phi_0(x) = Ae^{-x^2/x_0^2}.} \quad (7)$$

1.2 Higher energy eigenfunctions

Repeated application of the raising operator constructs the higher energy eigenfunctions:

$$\boxed{\phi_n(x) \propto (\hat{a}^\dagger)^n \phi_0(x).} \quad (8)$$

In the position representation,

$$\hat{a}^\dagger = \frac{x}{x_0} - \frac{\hbar}{p_0} \frac{d}{dx}, \quad x_0 p_0 = 2\hbar. \quad (9)$$

For the ground state,

$$\frac{d\phi_0}{dx} = -\frac{2x}{x_0^2} \phi_0. \quad (10)$$

Therefore

$$\hat{a}^\dagger \phi_0 = \left(\frac{x}{x_0} - \frac{\hbar}{p_0} \frac{d}{dx} \right) \phi_0 \quad (11)$$

$$= \left(\frac{x}{x_0} + \frac{2\hbar x}{p_0 x_0^2} \right) \phi_0 \quad (12)$$

$$= \frac{2x}{x_0} \phi_0. \quad (13)$$

Thus the first excited state has the form

$$\boxed{\phi_1(x) \propto x e^{-x^2/x_0^2}, \quad E_1 = E_0 + \hbar\omega = \frac{3}{2}\hbar\omega.} \quad (14)$$

Repeated differentiation and multiplication generate a polynomial multiplying the same Gaussian. These polynomials are the Hermite polynomials obtained earlier from the power-series method.

2 Degenerate eigenvalues

2.1 Eigenspaces and degeneracy

For an operator $\hat{O}$, the eigenspace belonging to eigenvalue o is

$$\mathcal{E}_o = \{\phi : \hat{O}\phi = o\phi\}. \quad (15)$$

The number of linearly independent eigenfunctions in this space is the degeneracy:

$$\boxed{g_o = \dim \mathcal{E}_o.} \quad (16)$$

Thus

$$\begin{aligned} g_o = 1 : & \quad o \text{ is nondegenerate,} \\ g_o > 1 : & \quad o \text{ is degenerate.} \end{aligned} \quad (17)$$

Multiplying an eigenfunction by a nonzero constant does not create a new linearly independent state.

Suppose two independent eigenfunctions have the same eigenvalue:

$$\hat{O}\phi_1 = o\phi_1, \quad \hat{O}\phi_2 = o\phi_2. \quad (18)$$

Linearity gives

$$\hat{O}(c_1\phi_1 + c_2\phi_2) = c_1\hat{O}\phi_1 + c_2\hat{O}\phi_2 \quad (19)$$

$$= o(c_1\phi_1 + c_2\phi_2). \quad (20)$$

Therefore every linear combination inside $\mathcal{E}_o$ has the same eigenvalue o .

2.2 Example: free-particle energy

For a free particle,

$$\hat{H} = \frac{\hat{p}^2}{2m}, \quad \phi_+(x) = e^{ikx}, \quad \phi_-(x) = e^{-ikx}. \quad (21)$$

The two functions have opposite momenta,

$$\hat{p}\phi_{\pm} = \pm\hbar k\phi_{\pm}, \quad (22)$$

but the same energy:

$$\hat{H}\phi_{\pm} = \frac{\hbar^2 k^2}{2m}\phi_{\pm}. \quad (23)$$

For $k \neq 0$, the two plane waves are linearly independent. The energy eigenvalue is therefore degenerate even though the momentum eigenvalues are different. These plane waves are generalized eigenfunctions rather than normalizable states on the full line.

2.3 Measurement of a degenerate eigenvalue

For an ideal measurement:

$$\begin{aligned} \text{nondegenerate } o : & \quad \text{state fixed up to an overall phase,} \\ \text{degenerate } o : & \quad \text{state known only to lie in } \mathcal{E}_o. \end{aligned} \quad (24)$$

Thus measuring a degenerate eigenvalue does not identify one unique basis eigenfunction. A second compatible observable can provide an additional label within the eigenspace.

Questions.

Question 1. A measurement of $\hat{O}$ gives a degenerate eigenvalue o . Does this result determine one unique eigenfunction of $\hat{O}$?

Answer and explanation. No. It determines only that the state lies in the eigenspace $\mathcal{E}_o$. Any normalized linear combination of basis eigenfunctions in that eigenspace has the same eigenvalue. Additional information, such as the result of measuring a compatible observable, is needed to select a unique state.

3 Common eigenstates and commutators

Suppose ϕ_{ab} is an eigenstate of both $\hat{A}$ and $\hat{B}$:

$$\hat{A}\phi_{ab} = a\phi_{ab}, \quad \hat{B}\phi_{ab} = b\phi_{ab}. \quad (25)$$

Then

$$[\hat{A}, \hat{B}]\phi_{ab} = (\hat{A}\hat{B} - \hat{B}\hat{A})\phi_{ab} \quad (26)$$

$$= (ab - ba)\phi_{ab} = 0. \quad (27)$$

Hence

$$\boxed{\text{common eigenstate} \implies [\hat{A}, \hat{B}]\phi_{ab} = 0.} \quad (28)$$

This statement concerns the particular state ϕ_{ab} . It is weaker than the operator identity $[\hat{A}, \hat{B}] = 0$, which must hold on every state. Under the usual spectral assumptions, commuting Hermitian operators can be represented using common eigenstates.

If

$$[\hat{A}, \hat{B}] = c\hat{I}, \quad c \neq 0, \quad (29)$$

then, for every nonzero state ϕ ,

$$[\hat{A}, \hat{B}]\phi = c\phi \neq 0. \quad (30)$$

Consequently, no common eigenstate can exist. In particular,

$$\boxed{[\hat{x}, \hat{p}] = i\hbar\hat{I}.} \quad (31)$$

Consequently, no nonzero state has both definite position and definite momentum.

Questions.

Question 2. If $[\hat{A}, \hat{B}] = c\hat{I}$ with $c \neq 0$, can a nonzero state be an eigenstate of both operators?

Answer and explanation. No. A common eigenstate would have to satisfy

$$[\hat{A}, \hat{B}]\phi = 0,$$

but the assumed commutator gives

$$[\hat{A}, \hat{B}]\phi = c\phi \neq 0$$

for every nonzero ϕ . The two conditions are incompatible.

4 The general uncertainty relation

4.1 Fluctuation operators and variances

For a normalized state ψ , define

$$\langle A \rangle = \int \psi^*(\hat{A}\psi) \, dx, \quad \boxed{\delta\hat{A} = \hat{A} - \langle A \rangle.} \quad (32)$$

The operator $\delta\hat{A}$ is the *fluctuation operator*; it measures the deviation of A from its expectation value. Since $\hat{A}$ is Hermitian and $\langle A \rangle$ is real,

$$(\delta\hat{A})^\dagger = \delta\hat{A}. \quad (33)$$

Its expectation value vanishes:

$$\langle \delta A \rangle = \langle A \rangle - \langle A \rangle = 0. \quad (34)$$

The variance is

$$(\Delta A)^2 = \int \psi^* (\delta\hat{A})^2 \psi \, dx \quad (35)$$

$$= \int (\delta\hat{A}\psi)^* (\delta\hat{A}\psi) \, dx \quad (36)$$

$$= \boxed{\int |\delta\hat{A}\psi|^2 \, dx}. \quad (37)$$

The second line follows from the Hermiticity of $\delta\hat{A}$.

4.2 Applying the Cauchy–Schwarz inequality

Apply Cauchy–Schwarz directly to $\delta\hat{A}\psi$ and $\delta\hat{B}\psi$:

$$(\Delta A)^2 (\Delta B)^2 = \left(\int |\delta\hat{A}\psi|^2 \, dx \right) \left(\int |\delta\hat{B}\psi|^2 \, dx \right) \quad (38)$$

$$\geq \left| \int (\delta\hat{A}\psi)^* (\delta\hat{B}\psi) \, dx \right|^2 \quad (39)$$

$$= \left| \int \psi^* \delta\hat{A} \delta\hat{B} \psi \, dx \right|^2. \quad (40)$$

Let

$$z = \int \psi^* \delta\hat{A} \delta\hat{B} \psi \, dx. \quad (41)$$

For any complex number z ,

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \geq |\operatorname{Im} z|^2. \quad (42)$$

Therefore

$$(\Delta A)^2 (\Delta B)^2 \geq |\operatorname{Im} z|^2. \quad (43)$$

4.3 Connecting the imaginary part to the commutator

Because both fluctuation operators are Hermitian,

$$z^* = \int \psi^* \delta\hat{B} \delta\hat{A} \psi \, dx. \quad (44)$$

Using $2i \operatorname{Im} z = z - z^*$,

$$2i \operatorname{Im} z = \int \psi^* (\delta\hat{A} \delta\hat{B} - \delta\hat{B} \delta\hat{A}) \psi \, dx \quad (45)$$

$$= \int \psi^* [\delta\hat{A}, \delta\hat{B}] \psi \, dx. \quad (46)$$

The expectation values are ordinary numbers and commute with every operator, so

$$[\delta\hat{A}, \delta\hat{B}] = [\hat{A} - \langle A \rangle, \hat{B} - \langle B \rangle] \quad (47)$$

$$= [\hat{A}, \hat{B}]. \quad (48)$$

Consequently,

$$\text{Im } z = \frac{1}{2i} \int \psi^* [\hat{A}, \hat{B}] \psi \, dx. \quad (49)$$

Combining the results and taking the positive square root gives the general uncertainty relation:

$$\Delta A \Delta B \geq \frac{1}{2} \left| \int \psi^* [\hat{A}, \hat{B}] \psi \, dx \right|. \quad (50)$$

4.4 Position and momentum

Using $[\hat{x}, \hat{p}] = i\hbar\hat{I}$ and normalization,

$$\int \psi^* [\hat{x}, \hat{p}] \psi \, dx = i\hbar \int |\psi|^2 \, dx \quad (51)$$

$$= i\hbar. \quad (52)$$

Therefore

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (53)$$

This uncertainty is not caused by imperfect instruments. It follows from the nonzero commutator and is a property of the quantum state itself.

Questions.

Question 3. What uncertainty relation follows from $[\hat{x}, \hat{p}] = i\hbar\hat{I}$ for a normalized state?

Answer and explanation. Substitution into the general relation gives

$$\Delta x \Delta p \geq \frac{1}{2} \left| i\hbar \int |\psi|^2 \, dx \right|.$$

Because $\int |\psi|^2 \, dx = 1$,

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

5 Boundary regularity

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m} \phi''(x) + V(x)\phi(x) = E\phi(x). \quad (54)$$

The behavior required at a boundary depends on the potential:

Potential near the boundary	Required behavior
$V(x)$ continuous	ϕ, ϕ', ϕ'' continuous
finite step in $V(x)$	ϕ, ϕ' continuous; ϕ'' may jump
$V(x) = -g\delta(x)$	ϕ continuous; ϕ' may jump

At a finite potential step, both ϕ and ϕ' remain continuous. A delta potential is singular, so its derivative jump must instead be found by integrating the Schrödinger equation across the singular point.

6 The attractive delta-function well

6.1 From a narrow well to a point interaction

Consider a narrow square well,

$$V_w(x) = \begin{cases} -g/w, & |x| < w/2, \\ 0, & |x| > w/2, \end{cases} \quad \int_{-\infty}^{\infty} V_w(x) dx = -g. \quad (55)$$

As its width approaches zero and its depth approaches infinity while the area remains fixed,

$$\boxed{w \rightarrow 0 \implies V_w(x) \rightarrow -g\delta(x), \quad g > 0.} \quad (56)$$

The minus sign makes this potential attractive. The Dirac delta is characterized by

$$\int_{-\infty}^{\infty} \delta(x) dx = 1, \quad \int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0). \quad (57)$$

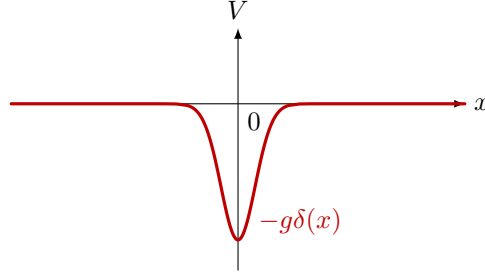


Figure 1: The downward spike schematically represents the attractive delta potential. The ideal potential is zero for $x \neq 0$ and has fixed area $-g$.

The stationary Schrödinger equation is

$$\boxed{\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - g\delta(x) \right] \phi(x) = E\phi(x).} \quad (58)$$

6.2 Bound-state solution away from the origin

The outside potential is $V_{\text{outside}} = 0$. A bound state must lie below this outside threshold:

$$\boxed{E < V_{\text{outside}} = 0.} \quad (59)$$

The negative sign of E is therefore a consequence of the chosen energy reference; it does not mean that the kinetic energy is negative.

For $x \neq 0$, $\delta(x) = 0$, so

$$-\frac{\hbar^2}{2m}\phi''(x) = E\phi(x). \quad (60)$$

Define the positive decay constant

$$\alpha \equiv \frac{\sqrt{-2mE}}{\hbar} = \frac{\sqrt{2m|E|}}{\hbar} > 0, \quad \ell = \frac{1}{\alpha}. \quad (61)$$

Then

$$E = -\frac{\hbar^2\alpha^2}{2m}, \quad \phi''(x) = \alpha^2\phi(x). \quad (62)$$

The length ℓ is the characteristic distance over which the wave function decays.

Normalizability removes the growing exponential on each side:

$$\phi(x) = \begin{cases} Ce^{\alpha x}, & x < 0, \\ De^{-\alpha x}, & x > 0. \end{cases} \quad (63)$$

Continuity at $x = 0$ gives $C = D = A$, so

$$\phi(x) = Ae^{-\alpha|x|}. \quad (64)$$

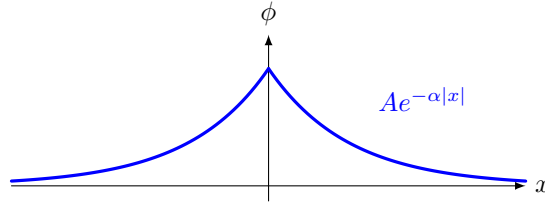


Figure 2: The bound-state wave function is continuous and localized near the attractive delta well, but its derivative changes discontinuously at the origin.

6.3 Derivative jump at the origin

Integrate the Schrödinger equation from $-\varepsilon$ to $+\varepsilon$:

$$-\frac{\hbar^2}{2m} \int_{-\varepsilon}^{\varepsilon} \phi'' dx - g \int_{-\varepsilon}^{\varepsilon} \delta(x)\phi(x) dx = E \int_{-\varepsilon}^{\varepsilon} \phi(x) dx. \quad (65)$$

As $\varepsilon \rightarrow 0$, the right-hand side vanishes, while the delta integral selects $\phi(0)$. Hence

$$-\frac{\hbar^2}{2m} [\phi'(0^+) - \phi'(0^-)] - g\phi(0) = 0, \quad (66)$$

or

$$\phi'(0^+) - \phi'(0^-) = -\frac{2mg}{\hbar^2}\phi(0). \quad (67)$$

For $\phi = Ae^{-\alpha|x|}$,

$$\phi'(0^+) = -\alpha A, \quad \phi'(0^-) = +\alpha A. \quad (68)$$

Substitution into the jump condition gives

$$-2\alpha A = -\frac{2mg}{\hbar^2}A, \quad \boxed{\alpha = \frac{mg}{\hbar^2}}. \quad (69)$$

Therefore the bound-state energy is

$$\boxed{E = -\frac{\hbar^2\alpha^2}{2m} = -\frac{mg^2}{2\hbar^2}}. \quad (70)$$

6.4 Normalization and physical meaning

Normalization determines A :

$$1 = \int_{-\infty}^{\infty} |\phi(x)|^2 dx \quad (71)$$

$$= 2|A|^2 \int_0^{\infty} e^{-2\alpha x} dx \quad (72)$$

$$= \frac{|A|^2}{\alpha}. \quad (73)$$

Choosing $A > 0$,

$$\boxed{\phi(x) = \sqrt{\alpha} e^{-\alpha|x|}, \quad \alpha = \frac{mg}{\hbar^2}}. \quad (74)$$

The jump condition fixes exactly one positive value of α , and therefore exactly one bound-state energy. An odd candidate would have $\phi(0) = 0$; continuity and exponential decay would then leave only the identically zero function. Thus

$$\boxed{\text{The attractive delta well has exactly one bound state.}} \quad (75)$$

Increasing g increases α , makes the state more localized, and makes its energy more negative relative to the outside threshold.

Questions.

Question 4. For $V(x) = -g\delta(x)$, what fixes the single allowed bound-state energy?

Answer and explanation. Normalizability first gives the decaying form $\phi = Ae^{-\alpha|x|}$. Integrating the Schrödinger equation across the origin gives the derivative jump condition. Substitution then fixes

$$\alpha = \frac{mg}{\hbar^2},$$

and therefore

$$\boxed{E = -\frac{mg^2}{2\hbar^2}}.$$

Because the jump condition selects only one positive α , the attractive delta well has only one bound state and no excited bound states.

Lecture 14 summary

- The raising operator constructs higher oscillator eigenfunctions:

$$\phi_n \propto (\hat{a}^\dagger)^n \phi_0, \quad \phi_1(x) \propto x e^{-x^2/x_0^2}.$$

- Degeneracy counts independent eigenfunctions that share one eigenvalue. A degenerate measurement fixes an eigenspace, not a unique basis eigenfunction.
- A common eigenstate satisfies $[\hat{A}, \hat{B}]\phi = 0$. Since $[\hat{x}, \hat{p}] = i\hbar\hat{I}$, no nonzero state has both definite position and definite momentum.
- The Cauchy–Schwarz inequality and the commutator give

$$\Delta A \Delta B \geq \frac{1}{2} \left| \int \psi^* [\hat{A}, \hat{B}] \psi \, dx \right|, \quad \Delta x \Delta p \geq \frac{\hbar}{2}.$$

- For the attractive delta well,

$$\phi'(0^+) - \phi'(0^-) = -\frac{2mg}{\hbar^2} \phi(0).$$

It has exactly one normalized bound state:

$$\phi(x) = \sqrt{\alpha} e^{-\alpha|x|}, \quad \alpha = \frac{mg}{\hbar^2}, \quad E = -\frac{mg^2}{2\hbar^2}.$$

Suggested reading. Griffiths and Schroeter, *Introduction to Quantum Mechanics*, 3rd ed., the sections on the algebraic treatment of the harmonic oscillator, the generalized uncertainty principle, and the delta-function potential. Review degenerate eigenspaces, the fluctuation operator, and the derivative jump condition.