

Lecture 13: Harmonic-Oscillator Ladder Operators and the Ground State

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Lecture 12 factorized the harmonic-oscillator Hamiltonian and introduced the adjoint of an operator. In this lecture we finish the adjoint examples, identify Hermitian operators, derive the ladder algebra, and use it to determine the complete oscillator energy spectrum and the ground-state wave function.

Questions.

- Why is d/dx anti-Hermitian while the momentum operator is Hermitian?
- Why must quantum-mechanical observables be represented by Hermitian operators?
- How do the commutators $[\hat{H}, \hat{a}]$ and $[\hat{H}, \hat{a}^\dagger]$ determine energy shifts?
- Why can the oscillator ladder not continue downward indefinitely?
- How does the operator method recover every allowed oscillator energy?
- How does $\hat{a}\phi_0 = 0$ determine the Gaussian ground state?

Lecture roadmap. We first complete the derivative and momentum adjoint calculations. We then derive the ladder commutators, follow their action on energy eigenstates, identify the bottom of the ladder, and solve the first-order equation for the ground state.

Review from Lecture 12

The oscillator Hamiltonian can be written as

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad (1)$$

where

$$\hat{a} = \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0}, \quad \hat{a}^\dagger = \frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0}, \quad (2)$$

and

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2\hbar m\omega}, \quad x_0 p_0 = 2\hbar. \quad (3)$$

The adjoint of $\hat{O}$ is defined by

$$\int f^* (\hat{O}^\dagger g) \, dx = \int (\hat{O} f)^* g \, dx. \quad (4)$$

For multiplication by a complex constant and for the position operator,

$$c^\dagger = c^*, \quad \hat{x}^\dagger = \hat{x}. \quad (5)$$

The derivative requires integration by parts, so its boundary term must be treated explicitly.

1 Adjoints, momentum, and Hermitian operators

1.1 The derivative operator

Let $\partial_x = d/dx$. Applying the definition of the adjoint gives

$$\int_{-\infty}^{\infty} f^* ((\partial_x)^\dagger g) \, dx = \int_{-\infty}^{\infty} (\partial_x f)^* g \, dx \quad (6)$$

$$= \int_{-\infty}^{\infty} (\partial_x f^*) g \, dx \quad (7)$$

$$= [f^* g]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f^* (\partial_x g) \, dx. \quad (8)$$

Complex conjugation commutes with differentiation because

$$\left(\frac{df}{dx} \right)^* = \frac{df^*}{dx}. \quad (9)$$

For oscillator wave functions in the chosen domain,

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} g(x) = 0, \quad (10)$$

so the boundary term vanishes. Therefore

$$\boxed{(\partial_x)^\dagger = -\partial_x.} \quad (11)$$

The derivative is *anti-Hermitian*: taking its adjoint produces a minus sign.

1.2 The momentum operator

In the position representation,

$$\hat{p} = -i\hbar \frac{d}{dx}. \quad (12)$$

Using $(c\hat{O})^\dagger = c^* \hat{O}^\dagger$,

$$\hat{p}^\dagger = \left(-i\hbar \frac{d}{dx} \right)^\dagger \quad (13)$$

$$= (+i\hbar) \left(-\frac{d}{dx} \right) \quad (14)$$

$$= -i\hbar \frac{d}{dx} = \hat{p}. \quad (15)$$

Thus

$$\boxed{\hat{p}^\dagger = \hat{p}.} \quad (16)$$

There are two sign changes: complex conjugation changes $-i$ to $+i$, and integration by parts changes d/dx to $-d/dx$. The signs cancel.

1.3 Hermitian operators and real eigenvalues

An operator is Hermitian on its domain if

$$\boxed{\hat{O}^\dagger = \hat{O}.} \quad (17)$$

Suppose $\phi_o \neq 0$ is an eigenfunction:

$$\hat{O}\phi_o = o\phi_o. \quad (18)$$

Multiply by ϕ_o^* and integrate:

$$\int \phi_o^* \hat{O}\phi_o \, dx = \int \phi_o^* (o\phi_o) \, dx \quad (19)$$

$$= o \int |\phi_o|^2 \, dx. \quad (20)$$

If $\hat{O}$ is Hermitian, the adjoint identity also gives

$$\int \phi_o^* \hat{O}\phi_o \, dx = \int (\hat{O}\phi_o)^* \phi_o \, dx \quad (21)$$

$$= o^* \int |\phi_o|^2 \, dx. \quad (22)$$

Since a nonzero eigenfunction has

$$\int |\phi_o|^2 \, dx > 0, \quad (23)$$

we obtain

$$o = o^*, \quad \boxed{o \in \mathbb{R}.} \quad (24)$$

Measurement outcomes are real numbers, and the possible outcomes of measuring an observable are its eigenvalues. Quantum mechanics therefore represents observables by Hermitian operators. In particular,

$$\hat{x}^\dagger = \hat{x}, \quad \hat{p}^\dagger = \hat{p}. \quad (25)$$

The ladder operator is different:

$$(\hat{a})^\dagger = \hat{a}^\dagger \neq \hat{a}. \quad (26)$$

Thus $\hat{a}$ and $\hat{a}^\dagger$ are useful mathematical operators, but neither one by itself represents an observable.

Questions.

Question 1. Assume the boundary term vanishes. What is the adjoint of d/dx ?

- A) d/dx
- B) $-d/dx$
- C) $i d/dx$

Answer and explanation. The answer is **B**. Integration by parts moves the derivative from the first function to the second and introduces a minus sign:

$$\int (\partial_x f)^* g \, dx = - \int f^* (\partial_x g) \, dx.$$

Hence $(\partial_x)^\dagger = -\partial_x$.

Questions.

Question 2. Is the lowering operator $\hat{a}$ itself an observable?

Answer and explanation. No. An observable must be represented by a Hermitian operator, whereas

$$\hat{a}^\dagger \neq \hat{a}.$$

The symbol $\hat{a}^\dagger$ is the adjoint of $\hat{a}$; it is a different operator.

2 The ladder algebra

2.1 The basic commutator

Using the definitions of the ladder operators,

$$[\hat{a}, \hat{a}^\dagger] = \left[\frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0}, \frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0} \right] \quad (27)$$

$$= -\frac{i}{x_0 p_0} [\hat{x}, \hat{p}] + \frac{i}{x_0 p_0} [\hat{p}, \hat{x}]. \quad (28)$$

Since

$$[\hat{x}, \hat{p}] = i\hbar, \quad [\hat{p}, \hat{x}] = -i\hbar, \quad x_0 p_0 = 2\hbar, \quad (29)$$

we find

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (30)$$

Consequently,

$$[\hat{a}^\dagger, \hat{a}] = -1, \quad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0. \quad (31)$$

Questions.

Question 3. Using $[\hat{a}, \hat{a}^\dagger] = 1$, calculate $[\hat{a}^\dagger, \hat{a}]$.

Answer and explanation. Interchanging the order of a commutator changes its sign:

$$[\hat{a}^\dagger, \hat{a}] = -[\hat{a}, \hat{a}^\dagger] = \boxed{-1}.$$

2.2 Commutators with the Hamiltonian

From

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad (32)$$

we obtain

$$[\hat{H}, \hat{a}] = \hbar\omega [\hat{a}^\dagger \hat{a}, \hat{a}] \quad (33)$$

$$= \hbar\omega (\hat{a}^\dagger \hat{a} \hat{a} - \hat{a} \hat{a}^\dagger \hat{a}) \quad (34)$$

$$= \hbar\omega [\hat{a}^\dagger, \hat{a}] \hat{a}. \quad (35)$$

Therefore

$$[\hat{H}, \hat{a}] = \boxed{-\hbar\omega \hat{a}}. \quad (36)$$

A similar calculation gives

$$\boxed{[\hat{H}, \hat{a}^\dagger] = +\hbar\omega\hat{a}^\dagger.} \quad (37)$$

These commutators are stronger than the statement that $\hat{a}$ and $\hat{a}^\dagger$ change a state label. They determine the exact change in energy.

2.3 Action on an energy eigenstate

Let

$$\hat{H}\phi_E = E\phi_E. \quad (38)$$

Using $\hat{H}\hat{a} = [\hat{H}, \hat{a}] + \hat{a}\hat{H}$,

$$\hat{H}(\hat{a}\phi_E) = ([\hat{H}, \hat{a}] + \hat{a}\hat{H})\phi_E \quad (39)$$

$$= (-\hbar\omega\hat{a} + E\hat{a})\phi_E \quad (40)$$

$$= (E - \hbar\omega)(\hat{a}\phi_E). \quad (41)$$

Provided $\hat{a}\phi_E \neq 0$, it is therefore an energy eigenfunction with energy $E - \hbar\omega$:

$$\boxed{\hat{a}\phi_E \propto \phi_{E-\hbar\omega}.} \quad (42)$$

Similarly,

$$\boxed{\hat{a}^\dagger\phi_E \propto \phi_{E+\hbar\omega}.} \quad (43)$$

The names are

$$\begin{aligned} \hat{a} : & \text{ lowering or annihilation operator,} \\ \hat{a}^\dagger : & \text{ raising or creation operator.} \end{aligned} \quad (44)$$

Here these operators remove or add one oscillator excitation of energy $\hbar\omega$. They do not destroy or create the particle itself.

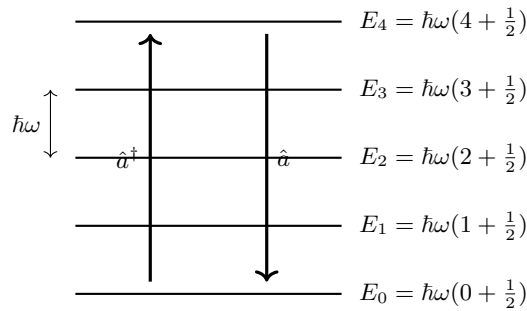


Figure 1: The ladder operators change the energy by one fixed quantum $\hbar\omega$.

Questions.

Question 4. If ϕ_E is an energy eigenfunction, what energy is associated with $\hat{a}^\dagger \phi_E$, provided this function is nonzero?

Answer and explanation. The raising commutator gives

$$\hat{H}(\hat{a}^\dagger \phi_E) = (E + \hbar\omega)(\hat{a}^\dagger \phi_E).$$

Thus

$$\hat{a}^\dagger \phi_E \text{ has energy } E + \hbar\omega.$$

The proportionality coefficient and normalization are not determined by the commutator alone.

3 The bottom of the ladder

3.1 A lower bound on the energy

For a normalized state ϕ ,

$$\int_{-\infty}^{\infty} |\phi(x)|^2 dx = 1. \quad (45)$$

Using the factorized Hamiltonian,

$$\int \phi^* \hat{H} \phi dx = \hbar\omega \left[\int \phi^* \hat{a}^\dagger \hat{a} \phi dx + \frac{1}{2} \int |\phi|^2 dx \right] \quad (46)$$

$$= \hbar\omega \left[\int (\hat{a}\phi)^* (\hat{a}\phi) dx + \frac{1}{2} \right] \quad (47)$$

$$= \hbar\omega \left[\int |\hat{a}\phi|^2 dx + \frac{1}{2} \right]. \quad (48)$$

The second equality uses the definition of the adjoint: moving $\hat{a}^\dagger$ to the other side of the integral changes it to $\hat{a}$. Since an absolute square cannot be negative,

$$\int \phi^* \hat{H} \phi dx \geq \frac{1}{2} \hbar\omega. \quad (49)$$

The lowering operation therefore cannot continue indefinitely. If it did, repeated energy shifts by $-\hbar\omega$ would eventually violate this lower bound.

3.2 The ground-state condition

At the bottom of the ladder, lowering must produce the identically zero function:

$$\hat{a}\phi_0 = 0. \quad (50)$$

The right-hand side is not a new physical state. It means there is no nonzero energy eigenfunction below ϕ_0 .

Acting with the Hamiltonian,

$$\hat{H}\phi_0 = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \phi_0 \quad (51)$$

$$= \frac{1}{2} \hbar\omega \phi_0. \quad (52)$$

Hence

$$\boxed{E_0 = \frac{1}{2} \hbar\omega.} \quad (53)$$

Questions.

Question 5. What must happen when the lowering operator reaches the ground state?

- A) It produces a physical state with negative energy.
- B) It produces the identically zero function: $\hat{a}\phi_0 = 0$.
- C) The commutator $[\hat{a}, \hat{a}^\dagger]$ becomes zero.

Answer and explanation. The answer is **B**. The commutator remains equal to one. The ladder stops because the lowering operator annihilates the ground state, so no nonzero state with a lower energy is produced.

3.3 All allowed oscillator energies

Start with any energy eigenstate ϕ_E . After n nonzero lowering steps,

$$\phi_b = \hat{a}^n \phi_E \neq 0, \quad E_b = E - n\hbar\omega. \quad (54)$$

Because the energy is bounded below, the lowering process must eventually stop:

$$\hat{a}\phi_b = 0. \quad (55)$$

Every state annihilated by $\hat{a}$ has the ground-state energy, so

$$E_b = \frac{1}{2} \hbar\omega. \quad (56)$$

Therefore

$$E - n\hbar\omega = \frac{1}{2} \hbar\omega, \quad (57)$$

and all allowed energies are

$$\boxed{E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots} \quad (58)$$

This reproduces the spectrum obtained from the power-series method, but now the equal spacing follows directly from the ladder commutators.

4 The ground-state wave function

The algebra gives both the ground-state energy and the equation

$$\hat{a}\phi_0 = 0. \quad (59)$$

Substitute

$$\hat{a} = \frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}, \quad \hat{p} = -i\hbar\frac{d}{dx}. \quad (60)$$

In the position representation,

$$\left(\frac{x}{x_0} + \frac{\hbar}{p_0}\frac{d}{dx}\right)\phi_0(x) = 0. \quad (61)$$

Using $x_0p_0 = 2\hbar$,

$$\boxed{\left(\frac{d}{dx} + \frac{2x}{x_0^2}\right)\phi_0(x) = 0.} \quad (62)$$

Separate the variables:

$$\frac{1}{\phi_0}\frac{d\phi_0}{dx} = -\frac{2x}{x_0^2}. \quad (63)$$

Integration gives

$$\ln \phi_0 = -\frac{x^2}{x_0^2} + C, \quad (64)$$

so

$$\boxed{\phi_0(x) = Ae^{-x^2/x_0^2}.} \quad (65)$$

Since

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}, \quad \frac{1}{x_0^2} = \frac{\alpha^2}{2}, \quad (66)$$

the same result can be written as

$$\boxed{\phi_0(x) = Ae^{-\alpha^2 x^2/2}.} \quad (67)$$

The operator method has reduced the original second-order Schrödinger equation to a first-order equation. Its solution is the same Gaussian ground state obtained by the direct method.

Lecture 13 summary

- With vanishing boundary terms,

$$(\partial_x)^\dagger = -\partial_x, \quad \hat{p}^\dagger = \hat{p}.$$

- Hermitian operators satisfy $\hat{O}^\dagger = \hat{O}$ and have real eigenvalues. Observables are represented by Hermitian operators.
- The oscillator ladder algebra is

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{H}, \hat{a}] = -\hbar\omega\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = +\hbar\omega\hat{a}^\dagger.$$

- Acting on an energy eigenstate,

$$\hat{a}\phi_E \propto \phi_{E-\hbar\omega}, \quad \hat{a}^\dagger\phi_E \propto \phi_{E+\hbar\omega}.$$

- Positivity of $\int |\hat{a}\phi|^2 dx$ gives

$$E \geq \frac{1}{2}\hbar\omega.$$

The ladder stops at $\hat{a}\phi_0 = 0$, giving

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right).$$

- The ground-state equation is first order and gives

$$\phi_0(x) = Ae^{-x^2/x_0^2} = Ae^{-\alpha^2 x^2/2}.$$

Suggested reading. Griffiths and Schroeter, *Introduction to Quantum Mechanics*, 3rd ed., the section on the algebraic treatment of the harmonic oscillator. Review the adjoint calculation, the ladder commutators, and the ground-state condition $\hat{a}\phi_0 = 0$.