

Lecture 12: Operator Methods for the Harmonic Oscillator

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Lecture 11 reduced the harmonic-oscillator equation to a power-series recurrence relation. In this lecture we impose normalizability, obtain the energy spectrum and Hermite eigenfunctions, and then begin a complementary operator method.

Questions.

- Why must the oscillator power series terminate?
- What are the energies, parity, and nodes of the oscillator eigenstates?
- What natural scales make the oscillator Hamiltonian dimensionless?
- How can the Hamiltonian be factorized into two operators?
- What is the adjoint of an operator, and how is it calculated?
- Why are the position and derivative operators treated differently under the adjoint?

Lecture roadmap. We finish the direct oscillator solution, interpret the first eigenstates, introduce natural scales and factorize the Hamiltonian, and then define the adjoint through three explicit examples.

Review from Lecture 11

The one-dimensional harmonic oscillator has

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \quad (1)$$

With

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x, \quad \lambda = \frac{2E}{\hbar\omega}, \quad (2)$$

the time-independent Schrödinger equation becomes

$$\psi''(\xi) + (\lambda - \xi^2)\psi(\xi) = 0. \quad (3)$$

Writing

$$\psi(\xi) = H(\xi)e^{-\xi^2/2} \quad (4)$$

gives the recurrence relation

$$a_{j+2} = \frac{2j+1-\lambda}{(j+2)(j+1)}a_j. \quad (5)$$

The recurrence separates into independent even and odd chains. Lecture 11 stopped before deciding which values of λ produce normalizable wave functions.

1 Normalizability and energy quantization

For large j , the recurrence relation approaches

$$\frac{a_{j+2}}{a_j} \approx \frac{2}{j}. \quad (6)$$

Compare this with

$$e^{\xi^2} = \sum_{r=0}^{\infty} \frac{\xi^{2r}}{r!}. \quad (7)$$

The ratio of consecutive coefficients in its even-power series is

$$\frac{1/(r+1)!}{1/r!} = \frac{1}{r+1} = \frac{2}{j+2} \approx \frac{2}{j}, \quad j = 2r. \quad (8)$$

Thus a nonterminating series eventually behaves as

$$H(\xi) \sim e^{\xi^2}. \quad (9)$$

The Gaussian factor that was removed earlier is then not sufficient to make the wave function decay:

$$\psi(\xi) = H(\xi)e^{-\xi^2/2} \sim e^{+\xi^2/2}. \quad (10)$$

Such a wave function is not square integrable. Therefore the physical series must terminate.

If a_n is the last nonzero coefficient, the numerator of the recurrence relation must vanish at $j = n$:

$$2n + 1 - \lambda = 0. \quad (11)$$

Consequently,

$$\boxed{\lambda_n = 2n + 1, \quad n = 0, 1, 2, \dots} \quad (12)$$

and, since $\lambda = 2E/(\hbar\omega)$,

$$\boxed{E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots} \quad (13)$$

For even n , the even coefficient chain terminates; for odd n , the odd chain terminates. Normalizability therefore selects both the parity and the allowed energy.

Questions.

Question 1. Why must the power series for $H(\xi)$ terminate?

- A) Otherwise H eventually overcomes the Gaussian decay and ψ is not normalizable.
- B) Otherwise the potential is not symmetric.
- C) Otherwise the energy becomes negative.

Answer and explanation. The answer is **A**. A nonterminating series behaves asymptotically like e^{ξ^2} , so $H e^{-\xi^2/2} \sim e^{+\xi^2/2}$ diverges. Termination converts H into a finite polynomial.

2 Hermite polynomials and eigenstates

The terminating polynomials are the Hermite polynomials H_n . The normalized energy eigenfunctions have the general form

$$\boxed{\psi_n(x) = \left(\frac{\alpha}{2^n n! \sqrt{\pi}} \right)^{1/2} H_n(\alpha x) e^{-\alpha^2 x^2/2}, \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}.} \quad (14)$$

The first Hermite polynomials are

$$H_0(\xi) = 1, \quad H_1(\xi) = 2\xi, \quad H_2(\xi) = 4\xi^2 - 2. \quad (15)$$

Therefore the first three normalized eigenfunctions are

$$\psi_0(x) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\alpha^2 x^2/2}, \quad (16)$$

$$\psi_1(x) = \frac{\sqrt{2\alpha}}{\pi^{1/4}} (\alpha x) e^{-\alpha^2 x^2/2}, \quad (17)$$

$$\psi_2(x) = \frac{1}{\pi^{1/4}} \sqrt{\frac{\alpha}{2}} (2\alpha^2 x^2 - 1) e^{-\alpha^2 x^2/2}. \quad (18)$$

The Gaussian controls the decay at large $|x|$, while the Hermite polynomial controls the parity and nodes. Since H_n has degree n ,

$$\boxed{\psi_n \text{ has } n \text{ interior nodes.}} \quad (19)$$

The parity alternates:

$$\psi_n(-x) = (-1)^n \psi_n(x). \quad (20)$$

The ground-state energy is not zero:

$$\boxed{E_0 = \frac{1}{2} \hbar \omega.} \quad (21)$$

This is the oscillator's *zero-point energy*. It is the energy of the ground state, not an additional state. The absence of a state with both $x = 0$ and $p = 0$ is consistent with $\Delta x \Delta p \geq \hbar/2$.

For the ground state, the classical turning points satisfy

$$\frac{1}{2} m \omega^2 x_{\pm}^2 = E_0, \quad \boxed{x_{\pm} = \pm \sqrt{\frac{\hbar}{m\omega}} = \pm \frac{1}{\alpha}.} \quad (22)$$

The probability density extends beyond these points. In fact,

$$P\left(|x| > \frac{1}{\alpha}\right) = \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-\xi^2} d\xi \approx 0.157. \quad (23)$$

Thus the ground state has about 15.7% probability in the classically forbidden region.

Questions.

Question 2. How many nodes does ψ_2 have, and what is its parity?

Answer and explanation. Since

$$\psi_2(\xi) \propto (2\xi^2 - 1) e^{-\xi^2/2},$$

its zeros occur at $\xi = \pm 1/\sqrt{2}$. The expression is unchanged under $\xi \rightarrow -\xi$, so

$$\boxed{\psi_2 \text{ has two nodes and is even.}}$$

Questions.

Question 3. What is the spacing between adjacent oscillator energies?

Answer and explanation. Using the energy spectrum,

$$E_{n+1} - E_n = \hbar\omega \left(n + 1 + \frac{1}{2} \right) - \hbar\omega \left(n + \frac{1}{2} \right) = \boxed{\hbar\omega}.$$

The direct calculation shows equal spacing but does not make its origin transparent. The operator method will explain this pattern algebraically.

Questions.

Question 4. What is the main motivation for introducing the operator method for the oscillator?

- A) To avoid normalizing wave functions.
- B) To explain the energy spacing and connect the eigenstates.
- C) To replace the Schrödinger equation by classical mechanics.

Answer and explanation. The answer is **B**. The power-series method remains correct, but factorization will show why neighboring energies differ by exactly $\hbar\omega$ and will connect neighboring eigenfunctions.

3 Natural scales

The oscillator contains the dimensional quantities $\hbar$, m , and ω . Their dimensions are

$$[\hbar] = ML^2T^{-1}, \quad [m] = M, \quad [\omega] = T^{-1}. \quad (24)$$

The natural energy scale is

$$E_{\text{scale}} = \hbar\omega, \quad (25)$$

because $[\hbar\omega] = ML^2T^{-2}$, the dimension of energy.

The natural length scale is obtained by combining $\hbar$, m , and ω to obtain dimensions of length:

$$\ell = \sqrt{\frac{\hbar}{m\omega}}. \quad (26)$$

The corresponding momentum scale is

$$p_{\text{scale}} = \frac{\hbar}{\ell} = \sqrt{\hbar m\omega}. \quad (27)$$

For convenient factorization, define slightly rescaled variables

$$\boxed{x_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2\hbar m\omega}.} \quad (28)$$

They satisfy

$$\boxed{x_0 p_0 = 2\hbar.} \quad (29)$$

4 Factorizing the Hamiltonian

Using $x_0^2 = 2\hbar/(m\omega)$ and $p_0^2 = 2\hbar m\omega$,

$$\frac{\hat{p}^2}{2m} = \hbar\omega \left(\frac{\hat{p}}{p_0} \right)^2, \quad \frac{1}{2}m\omega^2 \hat{x}^2 = \hbar\omega \left(\frac{\hat{x}}{x_0} \right)^2. \quad (30)$$

Therefore

$$\hat{H} = \hbar\omega \left[\left(\frac{\hat{p}}{p_0} \right)^2 + \left(\frac{\hat{x}}{x_0} \right)^2 \right]. \quad (31)$$

Consider the product

$$\left(\frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0} \right) \left(\frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0} \right) \quad (32)$$

$$= \frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2} + \frac{i}{x_0 p_0} [\hat{x}, \hat{p}]. \quad (33)$$

The canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$ and $x_0 p_0 = 2\hbar$ give

$$\frac{i}{x_0 p_0} [\hat{x}, \hat{p}] = \frac{i(i\hbar)}{2\hbar} = -\frac{1}{2}. \quad (34)$$

Thus, if we define

$$\hat{a} = \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0}, \quad \hat{a}^\dagger = \frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0}, \quad (35)$$

then

$$\hat{a}^\dagger \hat{a} = \left(\frac{\hat{x}}{x_0} \right)^2 + \left(\frac{\hat{p}}{p_0} \right)^2 - \frac{1}{2}. \quad (36)$$

Consequently,

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \quad (37)$$

The extra $1/2$ is a direct consequence of the noncommutativity of $\hat{x}$ and $\hat{p}$. If they commuted, the cross terms would cancel without leaving this term.

Questions.

Question 5. Why does the factorization produce the extra $1/2$?

- A) Because $\hat{x}$ and $\hat{p}$ commute.
- B) Because $[\hat{x}, \hat{p}] = i\hbar \neq 0$.
- C) Because the wave function has not been normalized.

Answer and explanation. The answer is **B**. Reordering the cross terms produces the commutator $[\hat{x}, \hat{p}]$, and

$$\frac{i}{x_0 p_0} [\hat{x}, \hat{p}] = -\frac{1}{2}.$$

This is why the Hamiltonian contains the additional zero-point term $\hbar\omega/2$.

5 Adjoints

For an operator $\hat{O}$, its adjoint $\hat{O}^\dagger$ is defined by

$$\int_{-\infty}^{\infty} f^*(x)(\hat{O}^\dagger g)(x) \, dx = \int_{-\infty}^{\infty} (\hat{O}f)^*(x)g(x) \, dx. \quad (38)$$

This definition says how the operator changes when it is moved from the first function to the second function in the integral. The equality must hold for all functions in the chosen domain.

Two useful rules are

$$(\hat{O}^\dagger)^\dagger = \hat{O}, \quad (\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger. \quad (39)$$

The order reverses in the adjoint of a product.

5.1 Example 1: a complex constant

Treat multiplication by $c \in \mathbb{C}$ as an operator:

$$c : f(x) \mapsto cf(x). \quad (40)$$

Then

$$\int f^*(c^\dagger g) \, dx = \int (cf)^* g \, dx \quad (41)$$

$$= \int c^* f^* g \, dx \quad (42)$$

$$= \int f^*(c^* g) \, dx. \quad (43)$$

Since this holds for every f and g ,

$$\boxed{c^\dagger = c^*}. \quad (44)$$

5.2 Example 2: the position operator

The position operator acts by multiplication:

$$(\hat{x}f)(x) = xf(x). \quad (45)$$

Because x is real,

$$\int f^*(\hat{x}^\dagger g) \, dx = \int (\hat{x}f)^* g \, dx \quad (46)$$

$$= \int (xf)^* g \, dx \quad (47)$$

$$= \int xf^* g \, dx \quad (48)$$

$$= \int f^*(\hat{x}g) \, dx. \quad (49)$$

Therefore

$$\boxed{\hat{x}^\dagger = \hat{x}}. \quad (50)$$

Questions.

Question 6. What does the dagger on $\hat{O}^\dagger$ mean?

- A) Complex conjugation of a wave function only.
- B) The adjoint operator defined by moving $\hat{O}$ across the integral.
- C) A measurement of the observable $\hat{O}$.

Answer and explanation. The answer is **B**. The dagger is an operation on an operator. It is defined by the integral identity above and is not itself a measurement.

Questions.

Question 7. If $c = 2 + i$ acts by multiplication, what is $c^\dagger$?

Answer and explanation. Complex conjugation gives

$$c^\dagger = c^* = 2 - i.$$

5.3 Example 3: the derivative operator

Let $\partial_x = d/dx$. Starting from the definition of the adjoint,

$$\int f^* ((\partial_x)^\dagger g) dx = \int (\partial_x f)^* g dx \quad (51)$$

$$= \int (\partial_x f^*) g dx \quad (52)$$

$$= [f^* g]_{-\infty}^{\infty} - \int f^* (\partial_x g) dx. \quad (53)$$

The last line is integration by parts. For normalizable oscillator states,

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} g(x) = 0, \quad (54)$$

so the boundary term vanishes:

$$[f^* g]_{-\infty}^{\infty} = 0. \quad (55)$$

Hence

$$\int f^* ((\partial_x)^\dagger g) dx = - \int f^* (\partial_x g) dx \quad (56)$$

$$= \int f^* (-\partial_x g) dx. \quad (57)$$

Therefore

$$(\partial_x)^\dagger = -\partial_x. \quad (58)$$

The boundary term matters: an adjoint depends not only on the formal differential expression but also on the operator domain and its boundary conditions. Here the oscillator wave functions vanish at infinity, so the displayed result applies.

Questions.

- Question 8.** Assume the boundary term vanishes. What is the adjoint of d/dx ?
- A) d/dx
 - B) $-d/dx$
 - C) $i d/dx$

Answer and explanation. The answer is **B**. Integration by parts moves the derivative to the second function and introduces a minus sign:

$$(\partial_x)^\dagger = -\partial_x.$$

Questions.

- Question 9.** Why must the boundary term be checked when finding an adjoint?
- A) Because it determines whether integration by parts produces an additional contribution.
 - B) Because it changes the value of $\hbar$.
 - C) Because it determines the number of spatial dimensions.

Answer and explanation. The answer is **A**. If the boundary term does not vanish, the formal expression $-\partial_x$ is not by itself the complete adjoint statement. The allowed functions and their boundary conditions are part of the operator definition.

Lecture 12 summary

- Normalizability forces the oscillator power series to terminate, giving

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad \psi_n(x) \propto H_n(\alpha x) e^{-\alpha^2 x^2/2}.$$

- The state ψ_n has parity $(-1)^n$, n interior nodes, and zero-point energy $E_0 = \hbar\omega/2$.
- The oscillator has natural scales $\hbar\omega$, $\ell = \sqrt{\hbar/(m\omega)}$, and $p_{\text{scale}} = \sqrt{\hbar m\omega}$.
- With $x_0 = \sqrt{2\hbar/(m\omega)}$ and $p_0 = \sqrt{2\hbar m\omega}$,

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad \hat{a} = \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0}.$$

- The adjoint is defined by

$$\int f^* (\hat{O}^\dagger g) \, dx = \int (\hat{O} f)^* g \, dx.$$

- The examples give

$$c^\dagger = c^*, \quad \hat{x}^\dagger = \hat{x}, \quad (\partial_x)^\dagger = -\partial_x,$$

where the last result uses the vanishing boundary term for oscillator states.

Lecture 12: Operator Methods for the Harmonic Oscillator

Suggested reading. Griffiths and Schroeter, *Introduction to Quantum Mechanics*, 3rd ed., the sections on the quantum harmonic oscillator and operator methods. Review the definition of an adjoint and the integration-by-parts calculation before Lecture 13.