

Lecture 11: Bound-State Structure and the Quantum Harmonic Oscillator

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Lecture 10 solved the finite square well through parity and boundary matching. We now extract general features of one-dimensional bound states and then apply them to a smooth potential, the quantum harmonic oscillator. For the oscillator, the goal of this lecture is to reduce the Schrödinger equation to a power-series recurrence relation.

Questions.

- How does the sign of $(E - V(x))$ determine the local form of an energy eigenfunction?
- How are the energies of one-dimensional bound states related to their nodes?
- What are the natural length and energy variables for the harmonic oscillator?
- Why is a Gaussian factor separated from the oscillator wave function?
- How does the power-series equation separate into even and odd solutions?

Lecture roadmap. We finish the general interpretation of finite-well bound states, introduce the harmonic-oscillator potential, make its Schrödinger equation dimensionless, determine the acceptable behavior at infinity, and derive the recurrence relation for the remaining power series.

Review from Lecture 10

The symmetric finite square well is

$$V(x) = \begin{cases} 0, & |x| < a/2, \\ V_0, & |x| \geq a/2, \end{cases} \quad 0 < E < V_0. \quad (1)$$

Inside the well, parity allows the eigenfunction to be written as

$$\phi_{\text{in}}(x) = \begin{cases} A \cos(kx), & \text{even state,} \\ B \sin(kx), & \text{odd state,} \end{cases} \quad k^2 = \frac{2mE}{\hbar^2}. \quad (2)$$

Outside the well, normalizability selects exponentially decaying tails with

$$\beta^2 = \frac{2m(V_0 - E)}{\hbar^2}. \quad (3)$$

Continuity of both ϕ and ϕ' at the finite walls gives

$$\boxed{\begin{aligned} k \tan(ka/2) &= \beta & (\text{even states}), \\ -k \cot(ka/2) &= \beta & (\text{odd states}). \end{aligned}} \quad (4)$$

Only special energies satisfy normalizability and the matching conditions. In the limit $V_0 \rightarrow \infty$, the tails disappear and the spectrum approaches

$$k_n = \frac{n\pi}{a}, \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}. \quad (5)$$

Questions.

Question 1. Which equation determines the even bound states of the symmetric finite square well?

- A) $k \tan(ka/2) = \beta$
- B) $-k \cot(ka/2) = \beta$
- C) $k = \beta$

Answer and explanation. The answer is **A**. An even interior solution is proportional to $\cos(kx)$. Matching its logarithmic derivative to the right-hand decaying tail at $x = a/2$ gives $k \tan(ka/2) = \beta$. The cotangent equation belongs to the odd sine solution.

Questions.

Question 2. What happens when $V_0 a^2$ increases?

- A) The graphical construction has more intersections.
- B) The well has fewer bound states.
- C) The energy spectrum becomes continuous.

Answer and explanation. The answer is **A**. The dimensionless well strength satisfies

$$\rho^2 = \frac{mV_0 a^2}{2\hbar^2}.$$

A larger $V_0 a^2$ gives a larger quarter circle in the graphical construction, allowing more intersections with the even and odd matching curves. Each intersection represents one bound-state energy.

1 Classically allowed and forbidden regions

For an energy eigenfunction,

$$-\frac{\hbar^2}{2m}\phi''(x) + V(x)\phi(x) = E\phi(x), \quad (6)$$

or

$$\phi''(x) + \frac{2m}{\hbar^2}[E - V(x)]\phi(x) = 0. \quad (7)$$

The sign of $E - V(x)$ determines the local behavior of the solution.

1.1 Classically allowed region

If $E > V(x)$, define

$$k^2 = \frac{2m[E - V(x)]}{\hbar^2} > 0. \quad (8)$$

For a constant potential, the local solution is oscillatory:

$$\boxed{E > V(x) \implies \phi(x) = A \cos(kx) + B \sin(kx).} \quad (9)$$

For the finite square well, this is the interior region $-a/2 < x < a/2$. A larger value of $E - V$ gives a larger k , a shorter wavelength, and faster oscillation.

1.2 Classically forbidden region

If $E < V(x)$, classical mechanics would require the kinetic energy

$$K = E - V(x) < 0, \quad (10)$$

which is impossible. This is why the region is called *classically forbidden*. Quantum mechanics does not require the wave function to vanish there. Instead, define

$$\beta^2 = \frac{2m[V(x) - E]}{\hbar^2} > 0. \quad (11)$$

For constant V , the local solutions are exponential:

$$E < V(x) \implies \phi(x) = Ae^{\beta x} + Be^{-\beta x}. \quad (12)$$

A bound-state wave function must satisfy

$$\int_{-\infty}^{\infty} |\phi(x)|^2 dx < \infty. \quad (13)$$

Therefore the exponentially growing branch must be discarded at each end of space. For the finite square well,

$$\phi(x) = \begin{cases} Ce^{\beta(x+a/2)}, & x < -a/2, \\ De^{-\beta(x-a/2)}, & x > a/2. \end{cases} \quad (14)$$

The wave function penetrates the classically forbidden region, but its physical tail decays at infinity.

Questions.

Question 3. At a point where $E > V(x)$, which behavior is expected locally?

- A) Oscillation
- B) Exponential decay only
- C) A constant wave function

Answer and explanation. The answer is **A**. When $E > V$, the coefficient $k^2 = 2m(E - V)/\hbar^2$ is positive, so the equation is $\phi'' + k^2\phi = 0$. Its independent solutions are sine and cosine.

Questions.

Question 4. For $x > a/2$, suppose $V_0 > E$ and

$$\phi(x) = Ae^{\beta x} + Be^{-\beta x}.$$

Which coefficient must vanish for a normalizable bound state as $x \rightarrow +\infty$?

- A) A
- B) B
- C) Neither

Answer and explanation. The answer is **A**. The term $e^{\beta x}$ diverges as $x \rightarrow +\infty$, making the probability integral infinite. The term $e^{-\beta x}$ decays and is the acceptable right-hand tail.

2 Allowed eigenstates and node ordering

Boundary conditions, matching, and normalizability select discrete energies

$$E_1 < E_2 < E_3 < \dots \quad (15)$$

For an ordinary connected one-dimensional potential, bound-state energies are nondegenerate. The eigenfunctions may be ordered by their number of interior nodes:

The n -th bound state has $n - 1$ interior nodes.

(16)

Thus the ground state has no interior node, and more nodes correspond to higher energy.

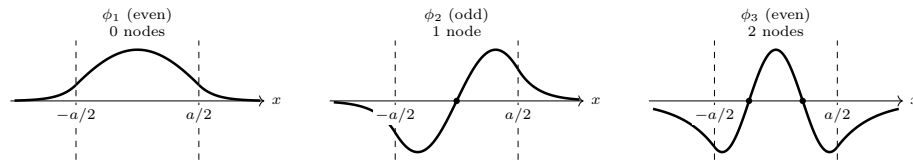


Figure 1: The first three bound-state eigenfunctions of one finite square well. The dots mark interior nodes; the values at the finite walls need not be zero.

The overall sign of an eigenfunction is arbitrary: ϕ_n and $-\phi_n$ represent the same physical state. The relevant information in Figure 1 is the parity, the nodes, and the decaying tails.

Questions.

Question 5. Why does a one-dimensional bound system normally have discrete energies?

- A) Every value of E gives a normalizable wave function.
- B) Only special values of E satisfy the boundary and normalizability conditions.
- C) Energy eigenfunctions must be plane waves.

Answer and explanation. The answer is **B**. The differential equation has solutions for many trial values of E , but most fail a boundary, matching, or normalizability condition. The special values satisfying all physical requirements form the discrete bound-state spectrum.

Questions.

Question 6. Three bound-state eigenfunctions have 0, 1, and 2 interior nodes. Their energies are E_A , E_B , and E_C , respectively. What is the usual ordering?

- A) $E_A < E_B < E_C$
- B) $E_A > E_B > E_C$
- C) $E_A = E_B = E_C$

Answer and explanation. The answer is **A**. In one dimension, the ground state has no interior node, the first excited state has one, and each higher state adds one node. The number of nodes therefore orders the bound-state energies.

3 The quantum harmonic oscillator

3.1 Hamiltonian and turning points

For a classical spring,

$$V(x) = \frac{1}{2}Kx^2, \quad F(x) = -\frac{dV}{dx} = -Kx. \quad (17)$$

Newton's equation becomes

$$\ddot{x} + \frac{K}{m}x = 0. \quad (18)$$

It is convenient to define the natural angular frequency

$$\omega^2 = \frac{K}{m}, \quad \omega = \sqrt{\frac{K}{m}}. \quad (19)$$

The quantum Hamiltonian is therefore

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \quad V(x) = \frac{1}{2}m\omega^2x^2. \quad (20)$$

Because the potential grows without bound as $|x| \rightarrow \infty$, a normalizable oscillator eigenfunction must approach zero at both ends.

For a given energy E , the classical turning points satisfy $E = V(x)$:

$$x_{\pm} = \pm\sqrt{\frac{2E}{m\omega^2}}. \quad (21)$$

Between the turning points, $E > V(x)$ and the eigenfunction is oscillatory. Outside them, $E < V(x)$ and a physical eigenfunction has decaying tails. The turning points are not hard walls, so the wave function need not vanish there.

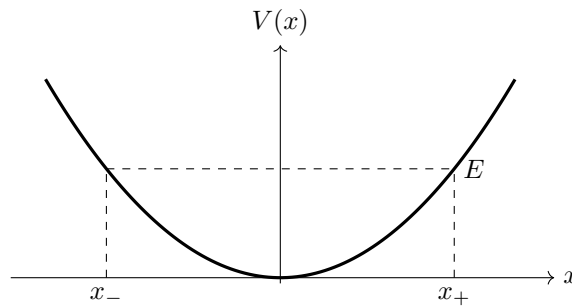


Figure 2: The harmonic-oscillator potential and the two turning points for an energy (E).

Questions.

Question 7. For the harmonic oscillator, where is the wave function oscillatory and where does it decay?

A) Oscillatory everywhere.

B) Exponential everywhere.

C) Oscillatory where $E > V(x)$ and decaying where $E < V(x)$.

Answer and explanation. The answer is **C**. The turning points divide space according to the sign of $E - V(x)$. Normalizability selects the decaying, rather than growing, solution in the two exterior regions.

Questions.

Question 8. For a particle of energy E in

$$V(x) = \frac{1}{2}m\omega^2 x^2,$$

find the classical turning points.

Answer and explanation. Setting $E = V(x)$ gives

$$E = \frac{1}{2}m\omega^2 x^2 \implies x^2 = \frac{2E}{m\omega^2}.$$

There are two symmetric turning points,

$$x_{\pm} = \pm \sqrt{\frac{2E}{m\omega^2}}.$$

4 Dimensionless Schrödinger equation

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x). \quad (22)$$

The constants m , ω , and $\hbar$ provide a natural inverse length

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}, \quad [\alpha] = L^{-1}. \quad (23)$$

We introduce the dimensionless coordinate and energy

$$\xi = \alpha x, \quad \lambda = \frac{2E}{\hbar\omega}. \quad (24)$$

Using

$$\frac{d}{dx} = \alpha \frac{d}{d\xi}, \quad \frac{d^2}{dx^2} = \alpha^2 \frac{d^2}{d\xi^2}, \quad (25)$$

Equation (22) becomes

$$-\frac{d^2\psi}{d\xi^2} + \xi^2 \psi = \lambda \psi, \quad (26)$$

or equivalently

$$\psi''(\xi) + (\lambda - \xi^2)\psi(\xi) = 0. \quad (27)$$

The rescaling removes all dimensional constants from the differential equation. Position is measured in units of $1/\alpha = \sqrt{\hbar/(m\omega)}$, and energy is measured in units of $\hbar\omega/2$.

Questions.

Question 9. If $E = \frac{5}{2}\hbar\omega$, what is λ ?

Answer and explanation. From the definition of dimensionless energy,

$$\lambda = \frac{2E}{\hbar\omega} = \frac{2(5\hbar\omega/2)}{\hbar\omega} = 5.$$

5 Behavior at large $|\xi|$

When $|\xi|$ is large, the ξ^2 term dominates the constant λ , so

$$\psi'' + (\lambda - \xi^2)\psi = 0 \implies \psi'' \approx \xi^2\psi. \quad (28)$$

Use the asymptotic form

$$\psi(\xi) \sim \xi^s e^{\gamma\xi^2/2}. \quad (29)$$

Its second derivative satisfies

$$\frac{\psi''}{\psi} = \gamma^2\xi^2 + \gamma(2s+1) + \frac{s(s-1)}{\xi^2}. \quad (30)$$

Matching the leading ξ^2 terms gives

$$\gamma^2 = 1, \quad \gamma = \pm 1. \quad (31)$$

Thus the two possible large- $|\xi|$ behaviors are

$$\psi(\xi) \sim A\xi^s e^{-\xi^2/2} + B\xi^s e^{+\xi^2/2}. \quad (32)$$

The second branch grows rapidly and is not square integrable. A bound-state solution must therefore have a Gaussian decay:

$$\psi(\xi) \sim \text{polynomial} \times e^{-\xi^2/2}. \quad (33)$$

Questions.

Question 10. Which asymptotic term must be absent from a bound-state wave function?

Answer and explanation. The term

$$B\xi^s e^{+\xi^2/2}$$

must be absent. Its modulus squared grows as e^{ξ^2} , so its integral over the real line diverges. The word “bound” is enforced mathematically by square integrability.

6 Removing the Gaussian factor

The asymptotic analysis suggests factoring out the known decay:

$$\psi(\xi) = H(\xi)e^{-\xi^2/2}. \quad (34)$$

This is an exact change of the unknown function, not an approximation. The derivatives are

$$\psi' = e^{-\xi^2/2}(H' - \xi H), \quad (35)$$

$$\psi'' = e^{-\xi^2/2}[H'' - 2\xi H' + (\xi^2 - 1)H]. \quad (36)$$

Substitution into Equation (27) cancels the Gaussian factor and gives

$$H'' - 2\xi H' + (\lambda - 1)H = 0. \quad (37)$$

Questions.

Question 11. After setting $\psi = He^{-\xi^2/2}$, which equation does H satisfy?

Answer and explanation. Differentiating the product twice and substituting into $\psi'' + (\lambda - \xi^2)\psi = 0$ gives

$$H'' - 2\xi H' + (\lambda - 1)H = 0.$$

The $\xi^2 H$ terms cancel because the Gaussian already contains the required large- $|\xi|$ behavior.

7 Power-series solution

Assume that H has a power-series expansion

$$H(\xi) = \sum_{j=0}^{\infty} a_j \xi^j. \quad (38)$$

Then

$$H'(\xi) = \sum_{j=0}^{\infty} (j+1)a_{j+1}\xi^j, \quad (39)$$

$$H''(\xi) = \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}\xi^j. \quad (40)$$

Substituting these series into Equation (37) and setting the coefficient of every power ξ^j to zero gives

$$(j+2)(j+1)a_{j+2} + (\lambda - 1 - 2j)a_j = 0. \quad (41)$$

Therefore

$$a_{j+2} = \frac{2j+1-\lambda}{(j+2)(j+1)} a_j. \quad (42)$$

Each coefficient is related to the coefficient two powers earlier. Consequently, the series separates into two independent chains:

$$a_0 \longrightarrow a_2 \longrightarrow a_4 \longrightarrow \cdots \quad (\text{even powers}), \quad (43)$$

$$a_1 \longrightarrow a_3 \longrightarrow a_5 \longrightarrow \cdots \quad (\text{odd powers}). \quad (44)$$

The two free coefficients a_0 and a_1 represent the two independent solutions expected for a second-order differential equation. For example,

$$a_2 = \frac{1-\lambda}{2} a_0, \quad a_3 = \frac{3-\lambda}{6} a_1. \quad (45)$$

Questions.

Question 12. Which coefficients determine the even and odd parts of H ?

Answer and explanation. The coefficient a_0 determines the entire even chain, while a_1 determines the entire odd chain. Because the recurrence changes j by two, even and odd powers never mix:

$$a_0 : \text{ even solution}, \quad a_1 : \text{ odd solution}.$$

Lecture 11 summary

- Where $E > V(x)$, an energy eigenfunction is locally oscillatory; where $E < V(x)$, its local form is exponential.
- Normalizability removes the growing exponential branches of a bound-state wave function.
- In an ordinary one-dimensional bound system, the states are ordered by their interior nodes: the n -th state has $n - 1$ nodes.
- The oscillator Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

- With $\xi = \sqrt{m\omega/\hbar}x$ and $\lambda = 2E/(\hbar\omega)$, the Schrödinger equation becomes

$$\psi'' + (\lambda - \xi^2)\psi = 0.$$

- Normalizability requires Gaussian decay at large $|\xi|$, motivating

$$\psi = H e^{-\xi^2/2}.$$

- The remaining function satisfies

$$H'' - 2\xi H' + (\lambda - 1)H = 0,$$

with recurrence relation

$$a_{j+2} = \frac{2j+1-\lambda}{(j+2)(j+1)} a_j.$$

- The recurrence separates into independent even and odd coefficient chains.

Suggested reading. Griffiths and Schroeter, *Introduction to Quantum Mechanics*, 3rd ed., sections on one-dimensional bound states and the quantum harmonic oscillator. Review Lecture 10 for the finite-square-well matching conditions and their graphical solutions.