

Quantum Mechanics Homework 6

Lectures 13–14

Instructions. Show the main steps of every calculation and briefly explain the physical meaning of your results. There are three problems, each worth 20 points. You may use the formulas below without deriving them.

Useful formulas

For the harmonic oscillator,

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad [\hat{a}, \hat{a}^\dagger] = 1, \quad (1)$$

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega \hat{a}^\dagger, \quad x_0 p_0 = 2\hbar, \quad (2)$$

with

$$\hat{a}^\dagger = \frac{x}{x_0} - \frac{\hbar}{p_0} \frac{d}{dx}. \quad (3)$$

The adjoint relation is

$$\int f^* (\hat{O}^\dagger g) dx = \int (\hat{O} f)^* g dx. \quad (4)$$

The general uncertainty relation is

$$\Delta A \Delta B \geq \frac{1}{2} \left| \int \psi^* [\hat{A}, \hat{B}] \psi dx \right|. \quad (5)$$

For the Dirac delta and a positive constant c ,

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0), \quad \int_0^{\infty} e^{-cx} dx = \frac{1}{c}. \quad (6)$$

Problem 1. Oscillator ladder and first excited state [20 points]

Let ϕ_n be a normalized harmonic-oscillator energy eigenfunction,

$$\hat{H}\phi_n = E_n\phi_n. \quad (7)$$

The normalized ground state is

$$\phi_0(x) = \left(\frac{2}{\pi x_0^2} \right)^{1/4} e^{-x^2/x_0^2}, \quad \hat{a}\phi_0 = 0. \quad (8)$$

- (a) Use $[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger$ to show that, when $\hat{a}^\dagger\phi_n \neq 0$, it is an energy eigenfunction with energy $E_n + \hbar\omega$. (5 points)

- (b) For an arbitrary normalized state ϕ , show that

$$\int \phi^* \hat{H} \phi dx = \hbar\omega \left[\int |\hat{a}\phi|^2 dx + \frac{1}{2} \right].$$

Explain why this gives a lower bound on the oscillator energy, and use $\hat{a}\phi_0 = 0$ to find E_0 . (5 points)

- (c) Apply $\hat{a}^\dagger$ explicitly to $\phi_0(x)$ and show that the normalized first excited state is

$$\phi_1(x) = \frac{2x}{x_0} \phi_0(x).$$

Verify its normalization using $[\hat{a}, \hat{a}^\dagger] = 1$, the adjoint relation, and $\hat{a}\phi_0 = 0$. Do not evaluate a Gaussian integral. (7 points)

- (d) State E_1 , the parity of ϕ_1 , and the number and location of its nodes. (3 points)

Problem 2. Degeneracy and uncertainty**[20 points]**

For a free particle on the full line,

$$\hat{p} = -i\hbar \frac{d}{dx}, \quad \hat{H} = \frac{\hat{p}^2}{2m}, \quad (9)$$

and consider the generalized eigenfunctions

$$\phi_+(x) = e^{ikx}, \quad \phi_-(x) = e^{-ikx}, \quad k > 0. \quad (10)$$

- (a) Calculate the momentum and energy eigenvalues of ϕ_+ and ϕ_- . Explain why the energy $E = \hbar^2 k^2 / (2m)$ is degenerate. (5 points)

- (b) Define

$$\chi(x) = \phi_+(x) + \phi_-(x) = 2 \cos(kx).$$

Show that χ is an energy eigenfunction with energy E , but is not a momentum eigenfunction. (5 points)

- (c) Show that $[\hat{H}, \hat{p}] = 0$. If an energy measurement gives $E = \hbar^2 k^2 / (2m)$, explain why this result alone does not identify a unique state and how a momentum measurement can distinguish ϕ_+ from ϕ_- . (5 points)

- (d) Use $[\hat{x}, \hat{p}] = i\hbar$ in the general uncertainty relation to derive $\Delta x \Delta p \geq \hbar/2$ for a normalized state. (5 points)

Problem 3. Attractive delta-well bound state [20 points]

Consider

$$V(x) = -g\delta(x), \quad g > 0, \quad (11)$$

with $V_{\text{outside}} = 0$. The stationary Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - g\delta(x) \right] \phi(x) = E\phi(x). \quad (12)$$

(a) Explain why a bound state must satisfy $E < V_{\text{outside}} = 0$. Define

$$\alpha = \frac{\sqrt{-2mE}}{\hbar} > 0$$

and solve the Schrödinger equation for $x \neq 0$. Use normalizability and continuity at the origin to obtain

$$\phi(x) = Ae^{-\alpha|x|}.$$

(5 points)

(b) Integrate the Schrödinger equation from $-\varepsilon$ to $+\varepsilon$, and take $\varepsilon \rightarrow 0$, to derive

$$\phi'(0^+) - \phi'(0^-) = -\frac{2mg}{\hbar^2}\phi(0).$$

(6 points)

(c) Apply the derivative jump condition to $\phi(x) = Ae^{-\alpha|x|}$. Find α and the bound-state energy E . (5 points)

(d) Normalize the bound-state wave function. Briefly explain why the attractive delta well has exactly one bound state and no excited bound states. (4 points)