

Quantum Mechanics Homework 5

Lectures 10–12

Instructions. Show the main steps of every calculation and briefly explain the physical meaning of your results. Each problem is worth 20 points. You may leave answers in terms of $\hbar$, m , ω , and the parameters given below.

Useful formulas

You may use the following formulas without deriving them:

$$\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}, \quad b > 0, \quad (1)$$

$$\int_{-\infty}^{\infty} x^2 e^{-bx^2} dx = \frac{\sqrt{\pi}}{2b^{3/2}}, \quad b > 0, \quad (2)$$

$$\int_{-\infty}^{\infty} x e^{-bx^2} dx = 0, \quad b > 0. \quad (3)$$

For a normalized wave function,

$$\langle F(x) \rangle = \int_{-\infty}^{\infty} \psi^*(x) F(x) \psi(x) dx. \quad (4)$$

For functions that vanish at infinity, integration by parts gives

$$\int_{-\infty}^{\infty} f^*(x) \frac{dg}{dx} dx = - \int_{-\infty}^{\infty} \left(\frac{df}{dx} \right)^* g(x) dx. \quad (5)$$

Problem 1. An even state in a finite square well [20 points]

Consider the symmetric finite square well

$$V(x) = \begin{cases} 0, & |x| < a/2, \\ V_0, & |x| \geq a/2, \end{cases} \quad 0 < E < V_0. \quad (6)$$

Define

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad \beta = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \quad (7)$$

For an even bound state, use

$$\phi(x) = \begin{cases} A \cos(kx), & |x| < a/2, \\ D e^{-\beta(x-a/2)}, & x \geq a/2, \end{cases} \quad (8)$$

with the left-hand tail fixed by even parity.

(a) Apply continuity of ϕ at $x = a/2$ and express D in terms of A . (4 points)

(b) Apply continuity of ϕ' at $x = a/2$. Divide the derivative equation by the result from part (a) to derive

$$k \tan\left(\frac{ka}{2}\right) = \beta.$$

(6 points)

(c) Suppose one graphical intersection occurs at

$$\xi = \frac{ka}{2} = \frac{\pi}{4}.$$

Use the matching condition to show that $\beta = k$, and then use

$$k^2 + \beta^2 = \frac{2mV_0}{\hbar^2}$$

to find the ratio E/V_0 . (6 points)

(d) The penetration length is $\ell = 1/\beta$. Find ℓ in terms of a for this state. State its parity and number of interior nodes. (4 points)

Problem 2. Power-series termination for the harmonic oscillator [20 points]

For the harmonic oscillator, define

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x, \quad \lambda = \frac{2E}{\hbar\omega}, \quad (9)$$

and write

$$\psi(\xi) = H(\xi)e^{-\xi^2/2}, \quad H(\xi) = \sum_{j=0}^{\infty} a_j \xi^j. \quad (10)$$

You may use the recurrence relation

$$a_{j+2} = \frac{2j+1-\lambda}{(j+2)(j+1)} a_j. \quad (11)$$

- (a) For the lowest even solution, require the series to terminate at $n = 0$. Find λ_0 , E_0 , and the form of $H_0(\xi)$. State the parity and number of nodes of ψ_0 . (5 points)
- (b) For the lowest odd solution, require the series to terminate at $n = 1$. Find λ_1 , E_1 , and the form of $H_1(\xi)$. State the parity and number of nodes of ψ_1 . (5 points)
- (c) For $n = 2$, find λ_2 and use the recurrence relation to calculate a_2 in terms of a_0 . Write $H_2(\xi)$ up to an overall constant, and determine the nodes of ψ_2 . (6 points)
- (d) Find the classical turning points for the $n = 2$ state by using

$$E_2 = \frac{1}{2}m\omega^2 x_{\pm}^2.$$

(4 points)

Problem 3. Gaussian ground state and expectation values [20 points]

Consider the harmonic-oscillator ground-state wave function

$$\psi_0(x) = Ae^{-\alpha^2 x^2/2}, \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}, \quad (12)$$

where A is real and positive. Use the Gaussian integrals listed at the beginning of the assignment.

- (a) Normalize ψ_0 and show that

$$A = \frac{\sqrt{\alpha}}{\pi^{1/4}}. \quad (5 \text{ points})$$

- (b) Find $\langle x \rangle$. Explain how the parity of the integrand allows you to obtain the result without evaluating a new integral. (4 points)

- (c) Calculate $\langle x^2 \rangle$ and the position uncertainty

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}.$$

Express your answers in terms of m , ω , and $\hbar$. (6 points)

- (d) Calculate the expectation value of the potential energy,

$$\langle V \rangle = \left\langle \frac{1}{2}m\omega^2 x^2 \right\rangle.$$

Given $E_0 = \hbar\omega/2$, infer the expectation value of the kinetic energy $\langle T \rangle$. (5 points)

Problem 4. Factorization and adjoints**[20 points]**

Define

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2\hbar m\omega}, \quad (13)$$

and use

$$[\hat{x}, \hat{p}] = i\hbar. \quad (14)$$

The adjoint of an operator is defined by

$$\int_{-\infty}^{\infty} f^*(x)(\hat{O}^\dagger g)(x) dx = \int_{-\infty}^{\infty} (\hat{O}f)^*(x)g(x) dx. \quad (15)$$

(a) Verify that $x_0 p_0 = 2\hbar$, and rewrite

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$$

as

$$\hat{H} = \hbar\omega \left[\left(\frac{\hat{p}}{p_0} \right)^2 + \left(\frac{\hat{x}}{x_0} \right)^2 \right]. \quad (5 \text{ points})$$

(b) Define

$$\hat{a} = \frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}, \quad \hat{a}^\dagger = \frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0}.$$

Expand $\hat{a}^\dagger \hat{a}$, keeping the order of $\hat{x}$ and $\hat{p}$, and show that

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \quad (6 \text{ points})$$

(c) Starting from the definition of the adjoint, show that the position operator satisfies

$$\hat{x}^\dagger = \hat{x}. \quad (4 \text{ points})$$

(d) Assume $f(x)$ and $g(x)$ vanish as $x \rightarrow \pm\infty$. Use integration by parts to show that

$$\left(\frac{d}{dx} \right)^\dagger = -\frac{d}{dx}.$$

Explain why the boundary condition is needed. (5 points)