

Quantum Mechanics Homework 4 Solutions

Lectures 8–9

Problem 1. Energy superposition and unitary evolution [20 points]

The orthonormality relations are

$$\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx = 1, \quad \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx = 1, \quad \int_{-\infty}^{\infty} \phi_1^*(x) \phi_2(x) dx = 0. \quad (1)$$

(a) Expanding the normalization integral gives

$$\int_{-\infty}^{\infty} |\psi(x, 0)|^2 dx = \frac{9}{25} \int |\phi_1|^2 dx + \frac{16}{25} \int |\phi_2|^2 dx \quad (2)$$

$$+ \frac{12i}{25} \int \phi_1^* \phi_2 dx - \frac{12i}{25} \int \phi_2^* \phi_1 dx \quad (3)$$

$$= \frac{9}{25} + \frac{16}{25} = 1. \quad (4)$$

The cross terms vanish because the two eigenfunctions are orthogonal.

(b) Each energy eigenfunction acquires its own phase:

$$\psi(x, t) = \frac{3}{5} \phi_1(x) e^{-iE_1 t/\hbar} + \frac{4i}{5} \phi_2(x) e^{-iE_2 t/\hbar}. \quad (5)$$

(c) The energy probabilities are the squared magnitudes of the expansion coefficients:

$$P(E_1) = \frac{9}{25}, \quad P(E_2) = \frac{16}{25}. \quad (6)$$

(d) At time t , the phase factors have magnitude one. Orthogonality still removes the cross terms in the integral, so

$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = \frac{9}{25} + \frac{16}{25} = 1. \quad (7)$$

Thus unitary evolution preserves normalization. The state is not stationary when $E_1 \neq E_2$ because the two components acquire different phases. Their relative phase changes with time, rather than the entire state changing by only one common phase.

Problem 2. Interference between two energies [20 points]

(a) Since ϕ_1 and ϕ_2 are real,

$$|\psi(x, t)|^2 = \frac{1}{2} [\phi_1^2(x) + \phi_2^2(x) + \phi_1(x)\phi_2(x)e^{-i(E_2-E_1)t/\hbar} + \phi_1(x)\phi_2(x)e^{i(E_2-E_1)t/\hbar}] \quad (8)$$

$$= \frac{1}{2} \left[\phi_1^2(x) + \phi_2^2(x) + 2\phi_1(x)\phi_2(x) \cos\left(\frac{(E_2 - E_1)t}{\hbar}\right) \right]. \quad (9)$$

(b) The coefficient ratio contains the relative phase

$$\frac{e^{-iE_2t/\hbar}}{e^{-iE_1t/\hbar}} = e^{-i(E_2-E_1)t/\hbar}. \quad (10)$$

Therefore the relative phase changes at angular frequency

$$\omega = \frac{|E_2 - E_1|}{\hbar}. \quad (11)$$

(c) One complete oscillation requires the phase to change by 2π :

$$T = \frac{2\pi\hbar}{|E_2 - E_1|}. \quad (12)$$

(d) The magnitudes of the two energy coefficients are both $1/\sqrt{2}$ and do not change with time. Hence

$$P(E_1) = P(E_2) = \frac{1}{2}. \quad (13)$$

The position density depends on the relative phase through the interference term, but an energy measurement depends only on the magnitudes of the energy-basis coefficients.

Problem 3. A square well with constant potential energy [20 points]

(a) Applying the first boundary condition gives

$$\phi(0) = A \cos(0) + B \sin(0) = A = 0. \quad (14)$$

Therefore,

$$\phi(x) = B \sin(kx). \quad (15)$$

(b) Applying the second boundary condition gives

$$\phi(L) = B \sin(kL) = 0. \quad (16)$$

Since $B \neq 0$ for a nonzero wavefunction,

$$kL = n\pi, \quad n = 1, 2, 3, \dots, \quad (17)$$

so

$$\boxed{k_n = \frac{n\pi}{L}}. \quad (18)$$

(c) Substituting $k_n = n\pi/L$ into the given energy formula gives

$$\boxed{E_n = V_0 + \frac{n^2\pi^2\hbar^2}{2mL^2}, \quad n = 1, 2, 3, \dots} \quad (19)$$

In particular,

$$\boxed{E_1 = V_0 + \frac{\pi^2\hbar^2}{2mL^2}}, \quad \boxed{E_2 = V_0 + \frac{4\pi^2\hbar^2}{2mL^2}}. \quad (20)$$

(d) Using the given normalization factor, the first two eigenfunctions inside the well are

$$\phi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right), \quad (21)$$

$$\phi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right). \quad (22)$$

Both are zero outside the well. The constant V_0 does not change the eigenfunctions or the allowed wavenumbers. It adds the same constant V_0 to every energy level:

$$E_n = E_n^{(0)} + V_0. \quad (23)$$

Problem 4. Infinite square well with symmetric boundaries [20 points]

Outside the well the wavefunction is zero. To use the familiar 0-to- L form of the infinite well, define

$$y = x + a. \quad (24)$$

Then $-a < x < a$ becomes $0 < y < 2a$.

(a) The boundary conditions become

$$\phi(0) = 0, \quad \phi(2a) = 0. \quad (25)$$

Since $dy/dx = 1$, the second derivative has the same form in the shifted coordinate. Inside the well,

$$-\frac{\hbar^2}{2m} \frac{d^2\phi}{dy^2} = E\phi, \quad (26)$$

or

$$\frac{d^2\phi}{dy^2} + k^2\phi = 0, \quad k^2 = \frac{2mE}{\hbar^2}. \quad (27)$$

The general solution is

$$\phi(y) = A \cos(ky) + B \sin(ky). \quad (28)$$

(b) The first boundary condition gives

$$\phi(0) = A = 0, \quad (29)$$

so $\phi(y) = B \sin(ky)$. The second boundary condition gives

$$\phi(2a) = B \sin(2ka) = 0. \quad (30)$$

For a nonzero wavefunction,

$$2ka = n\pi, \quad n = 1, 2, 3, \dots \quad (31)$$

Therefore,

$$\boxed{k_n = \frac{n\pi}{2a}} \quad (32)$$

and

$$\boxed{E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{8ma^2}}. \quad (33)$$

(c) Normalization requires

$$1 = |B|^2 \int_0^{2a} \sin^2\left(\frac{n\pi y}{2a}\right) dy = |B|^2 a. \quad (34)$$

Choosing B real and positive gives $B = 1/\sqrt{a}$. Returning to x ,

$$\boxed{\phi_n(x) = \frac{1}{\sqrt{a}} \sin \left[\frac{n\pi(x+a)}{2a} \right], \quad -a < x < a.} \quad (35)$$

The first three states may be written as

$$\phi_1(x) = \frac{1}{\sqrt{a}} \cos \left(\frac{\pi x}{2a} \right), \quad (36)$$

$$\phi_2(x) = -\frac{1}{\sqrt{a}} \sin \left(\frac{\pi x}{a} \right), \quad (37)$$

$$\phi_3(x) = -\frac{1}{\sqrt{a}} \cos \left(\frac{3\pi x}{2a} \right). \quad (38)$$

They are zero for $|x| \geq a$. The minus signs in ϕ_2 and ϕ_3 are only overall phases and have no physical effect.

- (d) From the expressions above, ϕ_1 is even, ϕ_2 is odd, and ϕ_3 is even. Their numbers of interior nodes are respectively 0, 1, and 2. Since $E_n \propto n^2$,

$$\boxed{E_1 : E_2 : E_3 = 1 : 4 : 9.} \quad (39)$$