

Quantum Mechanics – Homework 3 Solutions

Lectures 6–7

Instructor version. The main calculation and the physical meaning of each result are included.

Problem 1. Eigenstates and Measurement (20 points)

The orthonormality conditions, written in the position representation, are

$$\int \phi_i^*(x) \phi_j(x) dx = \delta_{ij}.$$

(a) Using the orthonormality integrals,

$$\begin{aligned} \int |\psi(x)|^2 dx &= \frac{3}{4} \int |\phi_1(x)|^2 dx + \frac{1}{4} \int |\phi_2(x)|^2 dx \\ &\quad + \frac{\sqrt{3}}{4} \int [\phi_1^*(x) \phi_2(x) + \phi_2^*(x) \phi_1(x)] dx \\ &= \frac{3}{4} + \frac{1}{4} = \boxed{1}. \end{aligned}$$

The cross terms vanish because $\int \phi_1^*(x) \phi_2(x) dx = 0$.

(b) The possible results are the eigenvalues a_1 and a_2 . Their probabilities are the squared moduli of the expansion coefficients:

$$\boxed{P(a_1) = \frac{3}{4}, \quad P(a_2) = \frac{1}{4}.}$$

(c) The expectation value is the probability-weighted average,

$$\boxed{\langle A \rangle = \frac{3}{4}a_1 + \frac{1}{4}a_2.}$$

This need not equal either individual measurement result.

(d) By linearity,

$$\begin{aligned} \hat{A}\psi &= \frac{\sqrt{3}}{2} \hat{A}\phi_1 + \frac{1}{2} \hat{A}\phi_2 \\ &= \boxed{\frac{\sqrt{3}}{2}a_1\phi_1 + \frac{1}{2}a_2\phi_2}. \end{aligned}$$

Since $a_1 \neq a_2$, this result is not one constant multiplied by ψ . Thus ψ is not an eigenstate of $\hat{A}$.

Calculating $\hat{A}\psi$ is a mathematical operation that produces another state-space vector. A measurement instead produces one numerical result, either a_1 or a_2 , with the probabilities found in part (b), and leaves the system in the corresponding eigenspace.

Problem 2. The Position–Momentum Commutator (20 points)

- (a) Acting first with $\hat{p}$ and then with $\hat{x}$ gives

$$\boxed{\hat{x}\hat{p}f(x) = -i\hbar xf'(x)}.$$

Reversing the order and using the product rule gives

$$\begin{aligned}\hat{p}\hat{x}f(x) &= -i\hbar \frac{d}{dx}[xf(x)] \\ &= \boxed{-i\hbar[f(x) + xf'(x)]}.\end{aligned}$$

- (b) Therefore

$$\begin{aligned}[\hat{x}, \hat{p}]f(x) &= \hat{x}\hat{p}f(x) - \hat{p}\hat{x}f(x) \\ &= -i\hbar xf'(x) + i\hbar[f(x) + xf'(x)] \\ &= \boxed{i\hbar f(x)}.\end{aligned}$$

Since this holds for every function in the common domain,

$$\boxed{[\hat{x}, \hat{p}] = i\hbar \hat{I}}.$$

- (c) Position and momentum do not commute, so they are incompatible observables. In particular, they cannot both have arbitrarily small uncertainties in the same state.
- (d) First convert the position uncertainty:

$$\Delta x = 0.20 \text{ nm} = 2.0 \times 10^{-10} \text{ m}.$$

The uncertainty relation gives

$$\begin{aligned}\Delta p &\geq \frac{\hbar}{2\Delta x} \\ &= \frac{1.055 \times 10^{-34} \text{ J s}}{2(2.0 \times 10^{-10} \text{ m})} \\ &= 2.64 \times 10^{-25} \text{ kg m/s}.\end{aligned}$$

Hence

$$\boxed{\Delta p_{\min} = 2.64 \times 10^{-25} \text{ kg m/s}}.$$

Problem 3. A Free-Particle Stationary State (20 points)

Here $\phi_k(x)$ is the same spatial energy eigenfunction denoted by $\phi_E(x)$ in Lecture 7. The subscript is changed from E to k because, for a free particle, $E_k = \hbar^2 k^2 / (2m)$.

(a) Since $|e^{ikx}|^2 = 1$,

$$\int_0^L |\phi_k(x)|^2 dx = \frac{1}{L} \int_0^L dx = \boxed{1}.$$

Also,

$$\hat{p}\phi_k = -i\hbar \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{L}} e^{ikx} \right) = \hbar k \phi_k, \quad \boxed{p_k = \hbar k}.$$

(b) Since $\partial_x^2 \phi_k = -k^2 \phi_k$,

$$\begin{aligned} \hat{H}\phi_k &= -\frac{\hbar^2}{2m} \frac{\partial^2 \phi_k}{\partial x^2} \\ &= \frac{\hbar^2 k^2}{2m} \phi_k, \end{aligned} \quad \boxed{E_k = \frac{\hbar^2 k^2}{2m} = \frac{p_k^2}{2m}}.$$

(c) An energy eigenstate evolves only by a phase:

$$\boxed{\psi(x, t) = \phi_k(x) e^{-iE_k t / \hbar} = \frac{1}{\sqrt{L}} e^{i(kx - E_k t / \hbar)}}.$$

Direct substitution gives

$$i\hbar \frac{\partial \psi}{\partial t} = E_k \psi, \quad \hat{H}\psi = e^{-iE_k t / \hbar} \hat{H}\phi_k = E_k \psi.$$

Thus ψ satisfies the time-dependent Schrödinger equation.

(d) The position probability density is

$$|\psi(x, t)|^2 = \frac{1}{L} |e^{ikx}|^2 |e^{-iE_k t / \hbar}|^2 = \boxed{\frac{1}{L}}.$$

It is independent of time, so the state is stationary even though the wave function has a time-dependent phase.

For ϕ_{-k} ,

$$p_{-k} = -\hbar k, \quad E_{-k} = \frac{\hbar^2 (-k)^2}{2m} = E_k.$$

Thus ϕ_k and ϕ_{-k} have opposite momenta but the same energy.